

# ON INTRINSIC GEOMETRY OF SURFACES IN NORMED SPACES

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**ABSTRACT.** We prove three facts about intrinsic geometry of surfaces in a normed (Minkowski) space. When put together, these facts demonstrate a rather intriguing picture. We show that (1) geodesics on saddle surfaces (in a space of any dimension) behave as they are expected to: they have no conjugate points and thus minimize length in their homotopy class; (2) in contrast, every two-dimensional Finsler manifold can be locally embedded as a saddle surface in a 4-dimensional space; and (3) geodesics on convex surfaces in a 3-dimensional space also behave as they are expected to: on a complete strictly convex surface, no complete geodesic minimizes the length globally.

## 1. INTRODUCTION

The goal of this paper is to prove three facts about intrinsic geometry of surfaces in a normed (Minkowski) space. When put together, these facts demonstrate a rather intriguing picture. Namely, Theorem 1.2 asserts that geodesics on saddle surfaces (in a space of any dimension) behave as they are expected to: they have no conjugate points and thus minimize length in their homotopy class. In contrast, Theorem 1.4 says that every two-dimensional Finsler manifold can be locally embedded as a saddle surface in a 4-dimensional normed space.

Thus the fact that geodesics on saddle surfaces minimize the length is global and, unlike in Riemannian geometry, it cannot be derived from studying local invariants such as the Gaussian curvature. Note that the property that a surface is saddle has nothing to do with various types of Finsler curvatures, for they can be negative or positive at some points of cylindrical surfaces.

Furthermore, Theorem 1.7 asserts that geodesics on convex surfaces (in a 3-dimensional space) also behave as they are expected to: on a complete strictly convex surface, no complete geodesic minimizes the length globally (and therefore some geodesics have conjugate points.) Therefore such a surface cannot be re-embedded as a saddle surface in any normed space (even though it can be re-embedded locally, hence this obstruction is of global nature). The nature of these phenomena remains obscure to us.

*Remark.* Interestingly enough, for *polyhedral* surfaces in normed spaces, global minimality of geodesics can be deduced from local intrinsic geometry: a globalization theorem holds. Studying Finsler geodesics has nice applications where there is no word “Finsler” in the formulation. For instance, consider a braid of several strings

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connecting two sets of nails in two parallel planes in  $\mathbb{R}^3$ . Having fixed topological type of the braid, one asks if the braid with the shortest total length of strings is unique (and if so, how convex is the length function near the optimum, compare with [2]). This question, having started from a purely Euclidean setup, naturally reduces to a problem about Finslerian geodesics. (We are grateful to Rahul [4] who brought this question to our attention.) We will address this aspect of geometry of polyhedral Finsler manifolds along with a few others elsewhere.

Now we proceed to definitions and formulations. Let  $\|\cdot\|$  be a norm on a finite dimensional vector space  $V$ . Note that the norm is uniquely determined by its unit ball  $B = \{v \in V : \|v\| \leq 1\}$  which is a centrally symmetric convex body in  $V$ . The boundary of  $B$  is the unit sphere of  $\|\cdot\|$ , it also determines the norm uniquely.

We say that a norm is  $C^r$ -smooth if it is a  $C^r$  function on  $V$  away from the origin. This is equivalent to the property that the unit sphere of the norm is a  $C^r$  hypersurface in  $V$ . If the  $C^r$  prefix is omitted, the term “smooth” means  $C^\infty$  (though the results are probably valid for  $C^2$ , we just did not care to chase the number of derivatives through the proofs).

A norm  $\|\cdot\|$  is said to be *strictly convex* if its unit sphere does not contain straight line segments. This is equivalent to the property that the triangle inequality

$$\|v + w\| \leq \|v\| + \|w\|, \quad v, w \in V$$

is strict unless  $v$  and  $w$  are proportional.

A norm  $\|\cdot\|$  on  $V$  is said to be *quadratically convex* if for every  $v \in V \setminus \{0\}$  there is a positive definite quadratic form on  $V$  whose square root majorizes the norm everywhere and equals the norm on the vector  $v$ . For smooth norms, this is equivalent to the following: the function  $\|\cdot\|$  has positive definite second derivative at every point of  $V \setminus \{0\}$ . Smooth quadratically convex norms are called *Minkowski norms*.

A (reversible) *Finsler metric* on a smooth manifold  $M$  is a continuous map  $\varphi: TM \rightarrow \mathbb{R}$  which is smooth away from the zero section and such that for every  $x \in M$  the restriction of  $\varphi$  to  $T_x M$  is a Minkowski norm. A *Finsler manifold* is a manifold  $M$  equipped with a Finsler metric. A detailed treatise of differential geometry of Finsler manifolds can be found e.g. in [1], below is a list of the basic definitions and facts that we use.

The length of a smooth curve  $\gamma: [a, b] \rightarrow M$  in a Finsler manifold  $M = (M, \varphi)$  is defined by

$$\text{length}(\gamma) = \int_a^b \varphi(\dot{\gamma}(t)) dt.$$

Geodesics in  $M$  are locally length minimizing curves. Equivalently, geodesics are critical points of the energy functional  $\gamma \mapsto \int \varphi^2(\dot{\gamma})$ , they are determined by the corresponding Euler–Lagrange equation. Smoothness and quadratic convexity of  $\varphi$  ensure that this equation is non-degenerate and imply the usual existence and uniqueness properties of solutions. All geodesics in this paper are assumed parameterized by arc length.

Surfaces in normed spaces are natural examples of Finsler manifolds. Namely if  $V$  is a vector space with a Minkowski norm  $\|\cdot\|$  and  $M$  is a smooth manifold, then every smooth immersion  $f: M \rightarrow V$  induces a Finsler  $\varphi$  metric on  $M$  given by  $\varphi(v) = \|df(v)\|$  for all  $v \in TM$ . If  $\varphi$ ,  $f$  and  $\|\cdot\|$  are so related, one also says that  $f$  is an *isometric immersion* of  $(M, \varphi)$  to  $(V, \|\cdot\|)$ .

**Definition 1.1.** A two-dimensional smooth surface  $S$  in  $\mathbb{R}^n$  (that is, a smooth immersion  $S: M \rightarrow \mathbb{R}^n$  where  $M$  is a two-dimensional manifold) is *strictly saddle* (resp. *saddle*) at a point  $p \in M$  if, for every normal vector at  $p$ , the second fundamental form of  $S$  with respect to this normal vector is indefinite (resp. indefinite or degenerate). A surface is (strictly) saddle if it is (strictly) saddle at every point.

One easily sees that this definition is affine invariant (or, equivalently, is independent of the Euclidean structure in the ambient space). Therefore it makes sense for surfaces in a vector space (without any Euclidean structure). In a Euclidean space, saddle surfaces have non-positive Gaussian curvature and therefore their geodesics have no conjugate points. Furthermore, only saddle surfaces preserve non-positiveness of curvature under all affine transformations, cf. [5].

The main result of this paper is the following theorem asserting that the “no conjugate points” property of saddle surfaces holds true in non-Euclidean normed spaces as well.

**Theorem 1.2.** *Let  $V$  be a finite dimensional space with a Minkowski norm and  $S$  a smooth saddle surface in  $V$ . Then every geodesic segment on  $S$  minimizes the length among all  $C^0$ -nearby curves with the same endpoints.*

The standard argument (similar to the proof of the Cartan–Hadamard theorem) shows that Theorem 1.2 implies the following.

**Corollary 1.3.** *Let  $M$  be a complete simply connected two-dimensional Finsler manifold which admits a saddle isometric immersion into a vector space with a Minkowski norm. Then every two points of  $M$  are connected by a unique geodesic, and all geodesics are length minimizers.*

These results could make one think that Finsler metrics of saddle surfaces have some special local properties (such as non-positivity of some curvature-like invariants) that imply this global properties. However the following theorem shows that this is not the case: every Finsler metric (including positively curved Riemannian metrics) can be locally realized as a metric of a saddle surface.

**Theorem 1.4.** *Let  $M$  be a two-dimensional Finsler manifold. Then every point of  $M$  has a neighborhood which admits a saddle smooth isometric embedding into a 4-dimensional normed space with a Minkowski norm.*

*Remark 1.5.* Every  $n$ -dimensional Finsler manifold can be locally isometrically embedded into a  $2n$ -dimensional normed space with a Minkowski norm, see [6] and references therein. Globally, every *compact* Finsler manifold  $M$  can be isometrically embedded in a finite dimensional normed space  $V$  but the dimension of  $V$  cannot be bounded above in terms of  $\dim M$  and moreover non-compact Finsler manifolds in general do not admit such embeddings, see [3].

*Remark 1.6.* It is still not clear whether saddle surfaces in 3-dimensional spaces are intrinsically different from convex ones. In other words, can a strictly saddle surface in a 3-dimensional normed space (with a Minkowski norm) be locally isometric to a strictly convex surface in another such space?

There might be obstructions to such isometries: it seems that, unlike in the Riemannian case, a generic Finsler metric does not admit any isometric embeddings into 3-dimensional spaces. So it would not be surprising that such an embedding, if it exists at all, is essentially unique.

The “opposite” to the class of saddle surfaces is the class of convex surfaces. Convex surfaces in  $\mathbb{R}^3$  are the only surfaces such that all their affine transformations are non-negatively curved, cf. [5]. The next theorem shows that geodesics on complete convex surfaces in normed spaces also possess properties typical for positive curvature.

**Theorem 1.7.** *Let  $V$  be a 3-dimensional normed space whose norm is  $C^1$ -smooth and strictly convex. Let  $B \subset V$  be a convex set with nonempty interior not containing straight lines (in other words,  $B$  is not a cylinder). Then there are no geodesic lines in  $\partial B$  (a geodesic line is a curve which is a shortest path between any pair of its points).*

The rest of the paper is organized as follows. Theorems 1.2, 1.4 and 1.7 are proved in sections 2, 3 and 4, respectively. These sections are completely independent from one another and each section introduces its own notation.

The proofs are mostly elementary although some parts involve cumbersome computations. We do not use any machinery of contemporary Finsler geometry (beyond things like the geodesic equation in Section 2). In fact, as shown by Theorem 1.4, this machinery would be useless here.

## 2. GEODESICS ON SADDLE SURFACES

The goal of this section is to prove Theorem 1.2.

**2.1. Preliminaries and notation.** We consider a finite dimensional vector space  $V$  with a Minkowski norm denoted by  $\Phi$ . As usual  $V^*$  denotes the dual space (that is the space of linear functions from  $V$  to  $\mathbb{R}$ ). By  $\langle \cdot, \cdot \rangle$  we denote the standard pairing between  $V^*$  and  $V$ , that is,  $\langle L, v \rangle = L(v)$  for  $L \in V^*$ ,  $v \in V$ .

The dual space  $V^*$  carries the dual norm  $\Phi^*$  given by

$$\Phi^*(L) = \sup\{\langle L, v \rangle : \Phi(v) = 1\},$$

this dual norm is also smooth and quadratically convex. The above supremum is attained at a unique vector from the unit sphere of  $\Phi$ , the direction of this vector is referred to as the direction of maximal growth, or the *gradient direction*, of  $L$ .

For a  $C^1$  function  $f: V \rightarrow \mathbb{R}$  and  $x \in V$ , we denote by  $df(x)$  the derivative of  $f$  at  $x$ . This is an element of  $V^*$ ; in our notation, the derivative of  $f$  at  $x$  along a vector  $v \in V$  is written as  $\langle df(x), v \rangle$ . If  $df(x) \neq 0$ , then the *gradient direction* of  $f$  at  $x$  is defined as that of the co-vector  $df(x)$ .

The *Legendre transform* of  $\Phi$  is the map  $\mathcal{L}_\Phi: V \rightarrow V^*$  defined by

$$\mathcal{L}_\Phi(v) = \frac{1}{2}d\Phi^2(v).$$

One easily sees that this map features the following properties:

- (i) it is positively homogeneous:  $\mathcal{L}_\Phi(tv) = t\mathcal{L}_\Phi(v)$  for all  $v \in V$  and  $t \geq 0$ ;
- (ii) if  $\Phi(v) = 1$ , then  $\mathcal{L}_\Phi(v)$  is the unique linear function  $L \in V^*$  such that  $\Phi^*(L) = 1$  and  $\langle L, v \rangle = 1$ ;
- (iii)  $\mathcal{L}_\Phi$  preserves the norm:  $\Phi^*(\mathcal{L}_\Phi(v)) = \Phi(v)$  for all  $v \in V$ ;
- (iv)  $\mathcal{L}_\Phi$  is a diffeomorphism between  $V \setminus \{0\}$  and  $V^* \setminus \{0\}$ , in particular, it is a diffeomorphism between the unit spheres of  $\Phi$  and  $\Phi^*$ ;
- (v) the inverse Legendre transform  $\mathcal{L}_\Phi^{-1}$  coincides with the Legendre transform  $\mathcal{L}_{\Phi^*}$  (as usual, we identify  $V^{**}$  with  $V$ ).

We use these properties without explicitly referring to them.

Let  $\gamma: I \rightarrow V$ , where  $I \subset \mathbb{R}$  is an interval, be a smooth unit-speed curve (that is,  $\Phi(\dot{\gamma}) \equiv 1$ ). The co-vector

$$K_\gamma(t) := \frac{d}{dt} \mathcal{L}_\Phi(\dot{\gamma}(t))$$

is referred to as the *curvature co-vector* of  $\gamma$  at  $t$ . (This co-vector takes the role of the curvature vector in the first variation formula.) A curve  $\gamma$  lying on a smooth submanifold  $M \subset V$  is a geodesic in  $M$  if and only if  $K_\gamma(t)$  annihilates the tangent space  $T_{\gamma(t)}M \subset V$  for all  $t$  (that is,  $\langle K_\gamma(t), v \rangle = 0$  for all  $v \in T_{\gamma(t)}M$ ).

For a Finsler metric  $\varphi$  on a manifold  $M$ , the notations  $\varphi^*$  and  $\mathcal{L}_\varphi$  denote the fiber-wise dual norm and the fiber-wise Legendre transform;  $\varphi^*$  is a function on  $T^*M$  and  $\mathcal{L}_\varphi$  is a map from  $TM$  to  $T^*M$ . Note that, if  $M \subset V$  is a smooth submanifold,  $\varphi$  is the induced Finsler metric on  $M$ ,  $x \in M$  and  $v \in T_xM$ , then  $\mathcal{L}_\varphi(v) = \mathcal{L}_\Phi|_{T_xM}$ .

**2.2. Calibrators.** Let  $S: M \rightarrow V$  be a saddle surface and  $\gamma: [a, b] \rightarrow M$  a geodesic of the induced Finsler metric  $\varphi$  on  $M$ . We are going to prove that  $\gamma$  minimizes length among  $C^0$ -nearby curves. It suffices to do this assuming that  $\gamma$  is embedded (that is, has no self-intersections in  $M$ ). Indeed, to reduce the general case to the special case when  $\gamma$  is embedded, construct an immersion

$$f: (a - \varepsilon, b + \varepsilon) \times (-\varepsilon, \varepsilon) \rightarrow M$$

such that  $f(t, 0) = \gamma(t)$  for all  $t \in [a, b]$  and apply the special case to the induced metric  $f^*\varphi$  on  $(a - \varepsilon, b + \varepsilon) \times (-\varepsilon, \varepsilon)$  and the geodesic  $t \mapsto (t, 0)$  there.

Throughout the rest of the proof we assume that  $\gamma$  is embedded and extended (as an embedded geodesic) to an interval  $(a - \varepsilon, b + \varepsilon)$ . We abuse notation and denote the image  $\gamma(a - \varepsilon, b + \varepsilon) \subset M$  by the same letter  $\gamma$ .

**Definition 2.1.** Let  $U \subset M$  be a neighborhood of  $\gamma([a, b])$ . A map  $h: U \rightarrow \mathbb{R}$  is said to be a *calibrator* for  $\gamma$  if the following holds:

- (i)  $h(\gamma(t)) = t$  for all  $t \in (a - \varepsilon, b + \varepsilon)$  such that  $\gamma(t) \in U$ ;
- (ii)  $\varphi^*(dh(x)) \leq 1$  for all  $x \in U$ .

If there is a calibrator for  $\gamma$  defined on a neighborhood  $U$ , then  $\gamma|_{[a, b]}$  is a unique shortest path in  $U$  between  $\gamma(a)$  and  $\gamma(b)$ . Indeed, let  $\gamma_1: [c, d] \rightarrow U$  be a piecewise smooth path with the same endpoints. Then

$$\begin{aligned} \text{length}(\gamma_1) &= \int_c^d \varphi(\dot{\gamma}_1(t)) dt \geq \int_c^d \langle dh(\gamma_1(t)), \dot{\gamma}_1(t) \rangle dt = \int_c^d \frac{d}{dt} h(\gamma_1(t)) dt \\ &= h(\gamma_1(d)) - h(\gamma_1(c)) = h(\gamma(b)) - h(\gamma(a)) = b - a = \text{length}(\gamma|_{[a, b]}). \end{aligned}$$

Here we used the fact that  $\varphi^*(dh(x)) \leq 1$  for all  $x \in U$  and hence  $\langle dh(x), v \rangle \leq \varphi(v)$  for all  $v \in T_xM$ .

**Definition 2.2.** Let  $U \subset M$  be a neighborhood of  $\gamma([a, b])$ . A map  $h: U \rightarrow \mathbb{R}$  is said to be an *almost calibrator* for  $\gamma$  if the following holds:

- (i)  $h(\gamma(t)) = t$  for all  $t \in (a - \varepsilon, b + \varepsilon)$  such that  $\gamma(t) \in U$ ;
- (ii)  $\varphi^*(dh(x)) \leq 1 + o(\text{dist}(x, \gamma)^2)$  as  $\text{dist}(x, \gamma) \rightarrow 0$ .

**Lemma 2.3.** *If  $\gamma$  admits an almost calibrator, then  $\gamma|_{[a, b]}$  is a shortest path in some neighborhood of its image.*

*Proof.* By the definition of almost calibrator, we have  $\langle dh(\gamma(t)), \dot{\gamma}(t) \rangle = 1$  and  $\varphi^*(dh(\gamma(t))) \leq 1$  for all  $t$ . Hence  $\varphi^*(dh(\gamma(t))) = 1$  and  $dh(\gamma(t)) = \mathcal{L}_\varphi(\dot{\gamma}(t))$ . We may assume that  $dh \neq 0$  on  $U$ .

Define a vector field  $V$  on  $U$  by

$$V(x) = \mathcal{L}_\varphi^{-1}(dh(x)) = \mathcal{L}_{\varphi^*}(dh(x)), \quad x \in U.$$

For any co-vector  $\xi \in T^*M$  such that  $\langle \xi, V(x) \rangle$ , the derivative of  $\varphi_x^*$  at  $dh(x) \in T^*M$  along  $\xi$  equals zero (this follows from the definition of the Legendre transform  $\mathcal{L}_{\varphi^*}$ ). Therefore

$$(2.1) \quad \varphi^*(dh(x) + \xi) \leq \varphi^*(dh(x)) + C\|\xi\|^2$$

for some constant  $C$ , all  $x \in U$  sufficiently close to  $\gamma([a, b])$ , and all  $\xi \in T^*M$  such that  $\langle \xi, V(x) \rangle = 0$ .

Recall that  $V(\gamma(t)) = \dot{\gamma}(t)$  for all  $t$ , so  $\gamma$  is a trajectory of  $V$ . Hence if  $U$  is sufficiently small, there is a smooth map  $f: U \rightarrow \mathbb{R}$  such that  $df \neq 0$  and  $f$  is constant along the trajectories of  $V$  or, equivalently,  $\langle df(x), V(x) \rangle = 0$  for all  $x \in U$ . We may assume that  $f = 0$  on  $\gamma$ , then

$$c \cdot \text{dist}(x, \gamma) \leq f(x) \leq C \cdot \text{dist}(x, \gamma)$$

for some constants  $c, C > 0$  and all  $x \in U$ . Define a function  $g: U \rightarrow \mathbb{R}$  by

$$g(x) = (1 - \sigma f(x)^2) \cdot h(x)$$

for a small  $\sigma > 0$ . Note that  $g = f$  on  $\gamma$ . We have

$$dg(x) = (1 - \sigma f(x)^2) \left( dh(x) - \frac{2\sigma f(x)}{1 - \sigma f(x)^2} \cdot df(x) \right).$$

Since  $\langle df(x), V(x) \rangle = 0$ , we can apply (2.1) to

$$\xi = -\frac{2\sigma f(x)}{1 - \sigma f(x)^2} \cdot df(x).$$

This yields

$$(2.2) \quad \varphi^*(dg(x)) \leq (1 - \sigma f(x)^2) \cdot \varphi^*(dh(x)) + C \cdot \frac{4\sigma^2 f(x)^2}{1 - \sigma f(x)^2} \cdot \|df(x)\|^2.$$

We may assume that  $U$  is so small that  $\sigma f(x)^2 < 1/2$  for all  $x \in U$  and  $\|df\|$  is bounded on  $U$ . Then the second summand in (2.2) is bounded above by  $C_1 \sigma^2 f(x)^2$  for some constant  $C_1 > 0$ . By the assumption (ii) of Definition 2.2, we have

$$\varphi^*(dh(x)) \leq 1 + o(\text{dist}(x, \gamma)^2) = 1 + o(f(x)^2), \quad \text{dist}(x, \gamma) \rightarrow 0.$$

Hence we have the following estimate for the first summand in (2.2):

$$(1 - \sigma f(x)^2) \cdot \varphi^*(dh(x)) \leq 1 - \frac{1}{2}\sigma f(x)^2$$

for all  $x$  sufficiently close to  $\gamma$ . Thus (2.2) implies that

$$\varphi^*(dg(x)) \leq 1 - \frac{1}{2}\sigma f(x)^2 + C_1 \sigma^2 f(x)^2 = 1 - \frac{1}{2}\sigma f(x)^2 (1 - 2C_1 \sigma)$$

for all  $x$  from a neighborhood  $U' \subset U$  of  $\gamma([a, b])$ . Hence  $\varphi^*(dg(x)) \leq 1$  for all  $x \in U'$  provided that  $\sigma < (2C_1)^{-1}$ . Thus  $g$  is a calibrator for  $\gamma$  in  $U'$ , therefore  $\gamma|_{[a, b]}$  is a shortest path in  $U'$ .  $\square$

**2.3. The construction.** Our goal is to construct an almost calibrator  $h$  for an embedded geodesic  $\gamma$  on our saddle surface. Recall that our surface is parameterized by  $S: M \rightarrow V$ . Let  $\gamma_S = S \circ \gamma$ . We define  $h: U \rightarrow \mathbb{R}$ , where  $U$  is a neighborhood of  $\gamma([a, b])$ , by the following implicit relation: the value  $h(x)$  is a parameter  $t \in (a - \varepsilon, b + \varepsilon)$  such that

$$(2.3) \quad \langle \mathcal{L}_\Phi(\dot{\gamma}_S(t)), S(x) - \gamma_S(t) \rangle = 0.$$

Observe that for  $x = \gamma(t)$  this equation is satisfied and the derivative of its left-hand side with respect to  $t$  is nonzero (more precisely, it equals  $-1$ ). Hence by the Implicit Function Theorem there exists a neighborhood  $U$  of  $\gamma$  and a unique smooth function  $h: U \rightarrow \mathbb{R}$  such that  $h(\gamma(t)) = t$  for all  $t$  and (2.3) holds for every  $x \in U$  and  $t = h(x)$ .

We are going to show that  $h$  is an almost calibrator for  $\gamma$ . The first requirement of Definition 2.2 is immediate from the construction. The second requirement is local; it suffices to verify it in a small neighborhood of every point of  $\gamma$ . Therefore we may assume that our surface is embedded and identify  $M$  with its image in the space. That is,  $M = U$  is a submanifold of  $V$  and  $S$  is the inclusion map  $M \rightarrow V$ . Then (2.3) takes the form

$$(2.4) \quad \langle \mathcal{L}_\Phi(\dot{\gamma}(t)), x - \gamma(t) \rangle = 0$$

where  $x \in M \subset V$ ,  $t = h(x)$ .

*Riemannian case.* Before proving that  $h$  is an almost calibrator, we briefly explain why this is true in the case when the ambient space is Euclidean. First observe that the condition (ii) in the definition of almost calibrator depends only on the derivatives of  $h$  at  $\gamma$  up to the second order. By (2.4), every level set  $h^{-1}(t)$  of  $h$  is the intersection of  $M$  with the hyperplane orthogonal to  $\gamma$  at  $\gamma(t)$ . This normal section of the surface has zero geodesic curvature at  $\gamma(t)$ , therefore it suffices to prove the result for a similar function whose level sets are geodesics orthogonal to  $\gamma$ . Since the Gaussian curvature of the surface is nonpositive, these geodesics diverge from one another, hence the distance between level sets is minimal at the base curve  $\gamma$ . This implies that the norm of the derivative of our function attains its minimum (equal to 1) at  $\gamma$ , hence the result.

#### 2.4. Computations.

**Lemma 2.4.** *Let  $x_0 = \gamma(t_0)$  where  $t_0 \in (a - \varepsilon, b + \varepsilon)$ . Then*

$$dh(x_0) = \mathcal{L}_\Phi(\dot{\gamma}(t_0))|_{T_{x_0}M} = \mathcal{L}_\varphi(\dot{\gamma}(t_0))$$

*and therefore  $\varphi^*(dh(x_0)) = 1$ .*

*Proof.* Recall that  $h(x_0) = t_0$ . By (2.4) we have

$$\langle \mathcal{L}_\Phi(\dot{\gamma}(h(x))), x - \gamma(h(x)) \rangle = 0$$

for all  $x \in M$ . Differentiate this identity at  $x = x_0$  along a vector  $v \in T_{x_0}M$ . Since the second term  $x - \gamma(h(x))$  of the above product is zero for  $x = x_0$ , the derivative of the first term cancels out, and the differentiation yields

$$\langle \mathcal{L}_\Phi(\dot{\gamma}(t_0)), v - \dot{\gamma}(t_0)h'_v \rangle = 0$$

where  $h'_v$  is the derivative of  $h$  at  $x_0$  along  $v$ , that is  $h'_v = \langle dh(x_0), v \rangle$ . Since  $\langle \mathcal{L}_\Phi(\dot{\gamma}(t_0)), \dot{\gamma}(t_0) \rangle = 1$ , it follows that  $h'_v = \langle \mathcal{L}_\Phi(\dot{\gamma}(t_0)), v \rangle$ . Since  $v$  is an arbitrary vector from  $T_{x_0}M$ , it follows that

$$dh(x_0) = \mathcal{L}_\Phi(\dot{\gamma}(t_0))|_{T_{x_0}M}.$$

Since  $\dot{\gamma}(t_0)$  is tangent to the surface, the right-hand side equals  $\mathcal{L}_\varphi(\dot{\gamma}(t_0))$ . The identity  $\varphi^*(dh(x_0)) = 1$  now follows from the fact that  $\varphi^*(\mathcal{L}_\varphi(v)) = \varphi(v)$  for every  $v \in TM$ .  $\square$

Now we introduce a special coordinate system  $(t, s)$  in a neighborhood of  $\gamma$ . The  $s$ -coordinate lines of this system are level curves of  $h$ . The  $t$ -coordinate lines are “gradient curves” of  $h$  (that is, curves tangent to the vector field  $\mathcal{L}_\varphi^{-1}(dh)$ ), in particular,  $\gamma$  itself is the  $t$ -coordinate line corresponding to  $s = 0$ .

More precisely, let  $r: (a-\varepsilon, b+\varepsilon) \times (-\varepsilon, \varepsilon) \rightarrow M \subset V$  be a local parameterization (whose argument is denoted by  $(t, s)$ ) such that for all  $(t, s)$  the following holds:

- (1)  $r(t, 0) = \gamma(t)$ ;
- (2)  $h(r(t, s)) = t$ ;
- (3) the first partial derivative  $r'_t$  of  $r$  at  $(t, s)$  is proportional to the vector  $\mathcal{L}_\varphi^{-1}(dh(x))$  where  $x = r(t, s)$ .

Lemma 2.4 ensures that these conditions are compatible. The third condition means that the vector  $r'_t$  points in the direction of the maximal growth of  $h$ . Since the derivative of  $h$  along this vector equals 1 (by the second condition), it follows that

$$\varphi^*(dh(x)) = \frac{1}{\varphi(r'_t(t, s))} = \frac{1}{\Phi(r'_t(t, s))}$$

for  $x = r(t, s)$ . Therefore the requirement (ii) of Definition 2.2 for  $h$  is equivalent to the following:

$$\Phi(r'_t(t, s)) \geq 1 - o(s^2), \quad s \rightarrow 0.$$

Denote

$$\rho(t, s) = \Phi^2(r'_t(t, s)).$$

Now it suffices to prove that

$$\rho(t, s) \geq 1 - o(s^2), \quad s \rightarrow 0.$$

By Lemma 2.4 we have  $\rho(t, 0) = 1$  for all  $t$ , therefore it suffices to prove that  $\rho'_s(t, 0) = 0$  and  $\rho''_{ss}(t, 0) \geq 0$  for all  $t$ .

Fix  $t_0 \in (a - \varepsilon, b - \varepsilon)$  and let us verify that  $\rho'_s(t_0, 0) = 0$  and  $\rho''_{ss}(t_0, 0) \geq 0$ . We introduce the following notation:

$$\begin{aligned} x_0 &= r(t_0, 0) = \gamma(t_0), \\ v(t, s) &= r'_t(t, s), \\ v_0 &= v(t_0, 0) = \dot{\gamma}(t_0), \\ L &= \mathcal{L}_\Phi(v_0) = \mathcal{L}_\Phi(\dot{\gamma}(t_0)), \\ K &= \left. \frac{d}{dt} \right|_{t=t_0} \mathcal{L}_\Phi(\dot{\gamma}(t)) \end{aligned}$$

Recall that  $K \in V^*$  is the “curvature co-vector” of  $\gamma$  at  $t_0$  and it annihilates  $T_x M$  (since  $\gamma$  is a geodesic). Using this notation, the definition of  $\rho$  can be written as

$$\rho(t, s) = \Phi^2(v(t, s)).$$



**Lemma 2.5.** *For all  $s \in (-\varepsilon, \varepsilon)$  we have*

$$(2.5) \quad \langle L, v'_s(t_0, s) \rangle = -\langle K, r'_s(t_0, s) \rangle,$$

$$(2.6) \quad \langle L, v'_s(t_0, 0) \rangle = 0,$$

$$(2.7) \quad \langle L, v''_{ss}(t_0, 0) \rangle = -\langle K, r''_{ss}(t_0, 0) \rangle.$$

*Proof.* The fact that  $h(r(t, s)) = t$  and (2.4) imply that

$$\langle \mathcal{L}_\Phi(\dot{\gamma}(t)), r(t, s) - r(t, 0) \rangle = 0$$

for all  $t, s$ . Differentiating this with respect to  $t$  yields

$$\left\langle \frac{d}{dt} \mathcal{L}_\Phi(\dot{\gamma}(t)), r(t, s) - r(t, 0) \right\rangle + \langle \mathcal{L}_\Phi(\dot{\gamma}(t)), r'_t(t, s) - r'_t(t, 0) \rangle = 0.$$

Since

$$\langle \mathcal{L}_\Phi(\dot{\gamma}(t)), r'_t(t, 0) \rangle = \langle \mathcal{L}_\Phi(\dot{\gamma}(t)), \dot{\gamma}(t) \rangle = 1,$$

it follows that

$$\left\langle \frac{d}{dt} \mathcal{L}_\Phi(\dot{\gamma}(t)), r(t, s) - r(t, 0) \right\rangle + \langle \mathcal{L}_\Phi(\dot{\gamma}(t)), r'_t(t, s) \rangle - 1 = 0,$$

or, equivalently,

$$\langle \mathcal{L}_\Phi(\dot{\gamma}(t)), v(t, s) \rangle = 1 - \left\langle \frac{d}{dt} \mathcal{L}_\Phi(\dot{\gamma}(t)), r(t, s) - r(t, 0) \right\rangle.$$

Substituting  $t = t_0$  and using the definitions of  $L$  and  $K$  yields

$$\langle L, v(t_0, s) \rangle = 1 - \langle K, r(t_0, s) - r(t_0, 0) \rangle.$$

Differentiating this with respect to  $s$  yields (2.5). Since  $r'_s(t_0, 0)$  is a tangent vector to  $M$  at  $x_0$ , we have  $\langle K, r'_s(t_0, 0) \rangle = 0$ , hence substituting  $s = 0$  into (2.5) yields (2.6). Finally, differentiating (2.5) with respect to  $s$  at  $s = 0$  yields (2.7).  $\square$

Recall that

$$L = \mathcal{L}_\Phi(v_0) = \frac{1}{2} d\Phi^2(v_0)$$

by the definitions of  $L$  and Legendre transform. Now we can verify that  $\rho'_s(t_0, 0) = 0$ :

$$\rho'_s(t_0, 0) = \left. \frac{d}{ds} \right|_{s=0} \Phi^2(v(t_0, s)) = \langle d\Phi^2(v_0), v'_s(t_0, 0) \rangle = 2\langle L, v'_s(t_0, 0) \rangle = 0$$

(the last identity follows from (2.6)).

Define a quadratic form  $Q$  on  $V$  by

$$Q = \frac{1}{2} d^2 \Phi^2(v_0)$$

(this is the second derivative at  $v_0$  of the function  $v \mapsto \frac{1}{2} \Phi^2(v)$  on  $V$ ). Since  $\Phi$  is a quadratically convex norm,  $Q$  is positive definite. We use  $Q$  as an auxiliary Euclidean structure on  $V$ .

From the definitions, for any  $w \in V$  we have

$$\langle K, w \rangle = \left\langle \frac{d}{dt} \right|_{t=t_0} \mathcal{L}_\Phi(\dot{\gamma}(t)), w \rangle = Q(\ddot{\gamma}(t_0), w)$$

since  $\mathcal{L}_\Phi(\dot{\gamma}(t)) = \frac{1}{2} d\Phi^2(\dot{\gamma}(t))$  and  $\dot{\gamma}(t_0) = v_0$ . In particular, the vector  $\ddot{\gamma}(t_0)$  is  $Q$ -orthogonal to the tangent plane  $T_{x_0}M$ . Let  $n$  be a  $Q$ -unit vector which is  $Q$ -orthogonal to  $T_{x_0}M$  and proportional to  $\ddot{\gamma}(t_0)$  if the latter is nonzero. Then

$$(2.8) \quad \langle K, w \rangle = Q(\ddot{\gamma}(t_0), w) = Q(\ddot{\gamma}(t_0), n) \cdot Q(w, n)$$

for every  $w \in V$ . Now we compute  $\rho''_{ss}(t_0, 0)$  as follows:

$$(2.9) \quad \frac{1}{2} \rho''_{ss}(t_0, 0) = \left. \frac{d^2}{ds^2} \right|_{s=0} \frac{1}{2} \Phi^2(v(t_0, s)) = Q(v'_s, v'_s) + L(v''_{ss})$$

where the partial derivatives  $v'_s$  and  $v''_{ss}$  are taken at  $(t_0, 0)$ . By (2.7), at  $(t, s) = (t_0, 0)$  we have

$$L(v''_{ss}) = -\langle K, r''_{ss} \rangle = -Q(\ddot{\gamma}(t_0), n) \cdot Q(r''_{ss}, n) = -Q(r''_{tt}, n) \cdot Q(r''_{ss}, n)$$

where the second identity follows from (2.8). Using this identity and the fact that  $v'_s = r''_{ts}$ , we rewrite (2.9) as follows:

$$\frac{1}{2}\rho''_{ss}(t_0, 0) = Q(r''_{ts}, r''_{ts}) - Q(r''_{tt}, n) \cdot Q(r''_{ss}, n).$$

With the trivial estimate  $Q(r''_{ts}, r''_{ts}) \geq Q(r''_{ts}, n)^2$ , this implies

$$\frac{1}{2}\rho''_{ss}(t_0, 0) \geq Q(r''_{ts}, n)^2 - Q(r''_{tt}, n) \cdot Q(r''_{ss}, n).$$

The right-hand side is minus the determinant of the second fundamental form of  $M$  with respect to the Euclidean structure  $Q$  and the normal vector  $n$ . Since  $M$  is a saddle surface, this determinant is nonpositive and we conclude that

$$\rho''_{ss}(t_0, 0) \geq 0.$$

As explained above, this inequality implies that  $h$  is an almost calibrator for  $\gamma$  and therefore (by Lemma 2.3)  $\gamma$  is a shortest path in a neighborhood of  $\gamma([a, b])$ . This finishes the proof of Theorem 1.2.

### 3. EXISTENCE OF SADDLE EMBEDDINGS

The goal of this section is to prove Theorem 1.4. Our plan is the following. First we define a saddle map  $F: U \rightarrow \mathbb{R}^4$ , where  $U$  is a small neighborhood of a point, and then we define a norm on  $\mathbb{R}^4$  such that  $F$  is an isometric embedding with respect to this norm. For such a norm to exist, the images of  $\varphi$ -unit vectors under  $dF$  should lie on a smooth strictly convex hypersurface in  $\mathbb{R}^4$  (this surface can be taken for the unit sphere of the norm that we want to construct). Our construction ensures that  $dF$  restricted to the set of  $\varphi$ -unit vectors parameterizes a strictly convex hypersurface located in a small neighborhood of a plane. Then a separate construction (described in the first subsection) is used to extend this surface to a compact smooth strictly convex hypersurface that can be taken for the unit sphere of a norm.

#### 3.1. Extending a convex surface.

**Definition 3.1.** Let  $\Sigma \subset \mathbb{R}^n$  be a smooth embedded hypersurface. We say that  $\Sigma$  is *pre-convex* if for every  $p \in \Sigma$  there is a linear function  $L: \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$(3.1) \quad L(q) \leq L(p) - c \cdot |p - q|^2$$

for some constant  $c > 0$  and all  $q \in \Sigma$ .

*Remark 3.2.* The function  $L$  satisfying (3.1) is unique up to multiplication by a constant: it must be zero on the tangent space  $T_p \Sigma \subset \mathbb{R}^n$ .

*Remark 3.3.* If (3.1) holds for all  $q$  close to  $p$ , then the second fundamental form of  $\Sigma$  at  $p$  (with respect to a suitable normal vector) is positive definite. Conversely, if the second fundamental form of  $\Sigma$  at  $p$  is positive definite, then (3.1) holds for all  $q$  from a sufficiently small neighborhood of  $p$ .

It follows that, if the requirement of Definition 3.1 is satisfied for all  $p$  from a compact set  $K \subset \Sigma$ , then some neighborhood of  $K$  in  $\Sigma$  is pre-convex.

**Lemma 3.4.** *Let  $\Sigma \subset \mathbb{R}^n$  be a pre-convex hypersurface and  $K \subset \Sigma$  a compact set. Then there exists a compact convex surface  $\Sigma'$  (that is, a boundary of a convex body) which is smooth, quadratically convex, and contains a neighborhood of  $K$  in  $\Sigma$ .*

*Furthermore, if  $\Sigma$  is symmetric with respect to the origin, then  $\Sigma'$  can be chosen symmetric as well.*

*Proof.* This is a standard type of argument, so we limit ourselves to a sketch. First of all, there is a neighborhood of  $K$  whose closure  $K_1$  is compact and contained in  $\Sigma$ . The most natural thing would be to take the convex hull of  $K_1$ , and the only problems would be that it is not necessarily smooth and quadratically convex.

It is easy to make it quadratically convex by taking the intersection  $B_1$  of all balls of radius  $R$  containing  $K_1$ , where  $R$  is larger than the reciprocal of normal curvatures of  $\Sigma$  over  $K_1$ . Then, by choosing  $\varepsilon > 0$  smaller than the reciprocal of normal curvatures of  $\Sigma$  over  $K_1$  and taking the inward  $\varepsilon$ -equidistant of the surface of  $B_1$  and then the outward  $\varepsilon$ -equidistant of the result, we obtain a surface of a body  $B_2$  which contains  $K_2$ , quadratically convex and  $C^1$ -smooth; furthermore, its principal curvatures are bounded between  $1/R$  and  $1/\varepsilon$  in the barrier sense.

All is left is to smoothen this surface further. This is done in a standard way by covering the surface by two open sets one of which contains  $K$  and the other does not intersect  $K$ . Then one approximates the radial function of  $B_2$  on the second set using convolutions. Sufficiently close approximations (with derivatives) will preserve quadratic convexity, and one concludes the argument by gluing these approximations with the original surface in a neighborhood of  $K$  using a partition of unity.  $\square$

**3.2. The case of constant metric.** For a Finsler metric  $\varphi$  in a region  $U \subset \mathbb{R}^2$  we denote by  $S_\varphi U$  the set of all  $\varphi$ -unit vectors in  $TU = U \times \mathbb{R}^2$ , that is,

$$S_\varphi U = \{v \in TU : \varphi(v) = 1\}.$$

Clearly  $S_\varphi U$  is a smooth 3-dimensional submanifold of  $TU$ .

We say that a Finsler metric on  $U \subset \mathbb{R}^2$  is *constant* if it does not depend on a point, that is there is a norm  $\|\cdot\|$  on  $\mathbb{R}^2$  such that  $\varphi(x, v) = \|v\|$  for all  $x \in U$ ,  $v \in T_x U \simeq \mathbb{R}^2$ . Of course this is a coordinate-dependent definition (though invariant under affine coordinate changes), however every flat Finsler metric locally admits a coordinate system in which it is constant.

**Lemma 3.5.** *For every constant Finsler metric  $\varphi$  on  $\mathbb{R}^2$  there exist a neighborhood  $U \subset \mathbb{R}^2$  of the origin and a smooth saddle embedding  $F: U \rightarrow \mathbb{R}^4$  such that the map  $dF|_{S_\varphi U}$  is an embedding and its image  $dF(S_\varphi U)$  is a pre-convex surface in  $\mathbb{R}^4$ .*

*Proof.* Let  $B$  be the unit ball of  $\varphi$  centered at 0 and  $S = \partial B$ . Then  $S_\varphi U = U \times S$  for any open set  $U \subset \mathbb{R}^2$ .

There is a parallelogram  $P$  containing  $B$  such that the midpoints of its four sides are on  $S$  (for example, consider a minimum area parallelogram containing  $B$ ). Introduce a new coordinate system  $(x, y)$  in the plane such that in these coordinates

$$P = \{(x, y) : x, y \in [-1, 1]\}.$$

Now  $B \subset P = [-1, 1]^2$  and  $B$  contains the four points  $(\pm 1, 0)$  and  $(0, \pm 1)$ .

For every  $\sigma > 0$ , define a map

$$F_\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$$

by

$$F_\sigma(x, y) = (f_\sigma(x, y), x^2 - y^2, xy)$$

where

$$f_\sigma(x, y) = (1 - \sigma^2 x^2 - \sigma^2 y^2) \cdot (x - \sigma x^3, y - \sigma y^3) \in \mathbb{R}^2.$$

Notice that  $F_\sigma$  converge to  $F_0$  as  $\sigma \rightarrow 0$ , where

$$F_0(x, y) = (x, y, x^2 - y^2, xy).$$

Observe that the derivative of  $F_\sigma$  at the origin is the standard inclusion of  $\mathbb{R}^2$  into  $\mathbb{R}^4$ :  $(\xi, \eta) \mapsto (\xi, \eta, 0, 0)$ . Therefore  $F_\sigma$  when restricted to a small neighborhood of the origin is a smooth embedding. We are going to show that  $F_\sigma$  satisfies the requirements for  $F$  for all sufficiently small  $\sigma > 0$ .

First we prove that, if  $\sigma$  is sufficiently small and  $U \subset \mathbb{R}^2$  is a sufficiently small neighborhood of the origin, then  $F_\sigma|_U$  is strictly saddle and  $dF_\sigma|_{S_\varphi U}$  is an embedding. Since  $F_\sigma$  converges to  $F_0$  with the derivatives as  $\sigma \rightarrow 0$ , it suffices to verify these facts for  $F_0$ .

Let us show that  $F_0$  is strictly saddle at the origin (by continuity of the second fundamental form, this implies that it is strictly saddle near the origin). For a unit vector  $\nu$  normal to  $F_0$  at the origin, denote by  $Q_\nu$  the second fundamental form of  $F_0$  with respect to  $\nu$ . A unit normal vector  $\nu$  can be written as  $\nu = \alpha e_3 + \beta e_4$  where  $\alpha^2 + \beta^2 = 1$ . Then  $Q_\nu = \alpha Q_{e_3} + \beta Q_{e_4}$ , and the quadratic forms  $Q_{e_3}$  and  $Q_{e_4}$  are given by

$$Q_{e_3}(x, y) = x^2 - y^2, \quad Q_{e_4}(x, y) = xy$$

for all  $x, y \in \mathbb{R}$ . The forms  $Q_{e_3}$  and  $Q_{e_4}$  are linearly independent, hence  $Q_\nu \neq 0$ . Furthermore, since  $Q_{e_3}$  and  $Q_{e_4}$  are traceless, so is  $Q_\nu$ , and thus  $Q_\nu$  is indefinite. Hence  $F_0$  is saddle at 0.

Now we show that  $dF_0|_{S_\varphi U}$  is an embedding provided that  $U \subset \mathbb{R}^2$  is a sufficiently small neighborhood of 0. For brevity, we denote  $dF_0: T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$  by  $G$ . In coordinates,  $G$  is given by

$$G(x, y, \xi, \eta) = (\xi, \eta, 2(x\xi - y\eta), 2(x\eta + y\xi))$$

where  $(x, y)$  are coordinates in  $\mathbb{R}^2$  and  $(\xi, \eta)$  are coordinates in  $T_{(x, y)}\mathbb{R}^2$ .

Recall that  $S_\varphi U = U \times S$  and observe that  $dF_0|_{\{0\} \times S}$  is injective. Therefore it suffices to verify that the partial derivatives of  $G$  at every point of  $\{0\} \times S$  are linearly independent. And this is trivial because

$$\begin{aligned} \frac{\partial G}{\partial x}(x, y, \xi, \eta) &= (0, 0, 2\xi, 2\eta), \\ \frac{\partial G}{\partial y}(x, y, \xi, \eta) &= (0, 0, -2\eta, 2\xi), \\ \frac{\partial G}{\partial \xi}(x, y, \xi, \eta) &= (1, 0, 2x, 2y), \\ \frac{\partial G}{\partial \eta}(x, y, \xi, \eta) &= (0, 1, -2y, 2x), \end{aligned}$$

so  $\det(dG) = \xi^2 + \eta^2$ , and  $(\xi, \eta) \neq (0, 0)$  if  $(\xi, \eta) \in S$ .

It remains to show that the set  $\Sigma := dF_\sigma(S_\varphi U) = dF_\sigma(U \times S)$  is pre-convex for some  $\sigma > 0$  and some neighborhood  $U \subset \mathbb{R}^2$  of the origin. We are going to show that for every sufficiently small  $\sigma$  there exists  $U$  such that  $\Sigma$  is pre-convex. In other words, we assume that  $\sigma \ll 1$  and  $|x|, |y| \ll \sigma$  for all  $(x, y) \in U$ . By Remark 3.3, it

suffices to verify that the requirement of Definition 3.1 is satisfied for every point  $p \in dF_\sigma(\{0\} \times S)$ .

Let  $p = dF_\sigma(0, 0, \xi_0, \eta_0) = (\xi_0, \eta_0, 0, 0)$  where  $v_0 := (\xi_0, \eta_0) \in S$ . Let  $L_0: \mathbb{R}^2 \rightarrow \mathbb{R}$  be the supporting linear function for  $B$  at  $v_0$ , that is,  $L_0(v) \leq 1$  for all  $v \in B$  and  $L_0(v_0) = 1$ . Since  $\varphi$  is a quadratically convex norm, we have

$$(3.2) \quad L_0(v) \leq 1 - c_0 \cdot |v - v_0|^2$$

for some  $c_0 > 0$  and all  $v \in B$ . Define  $L: \mathbb{R}^4 \rightarrow \mathbb{R}$  by

$$L(x, y, z, t) = L_0(x, y).$$

We are going to show that  $L$  satisfies (3.1) for all  $q \in dF_\sigma(U \times S)$ . Since we have already verified that  $dF_\sigma|_{U \times S}$  is a smooth embedding, it suffices to show that

$$L(dF_\sigma(x, y, \xi, \eta)) \leq 1 - c \cdot (x^2 + y^2 + (\xi - \xi_0)^2 + (\eta - \eta_0)^2)$$

for some  $c > 0$  and all  $(x, y) \in U$ ,  $(\xi, \eta) \in S$ . Note that

$$L(dF_\sigma(x, y, \xi, \eta)) = L_0(df_\sigma(x, y, \xi, \eta))$$

by the definitions of  $L$  and  $F_\sigma$ , so we need to show that

$$(3.3) \quad L_0(df_\sigma(x, y, \xi, \eta)) \leq 1 - c \cdot (x^2 + y^2 + (\xi - \xi_0)^2 + (\eta - \eta_0)^2)$$

for some  $c > 0$ .

Since the definition of  $df_\sigma$  is symmetric with respect to the changes  $x \mapsto -x$ ,  $y \mapsto -y$  and  $x \leftrightarrow y$ , it suffices to consider the case when  $\xi_0 \geq \eta_0 \geq 0$ . Since  $B$  is inscribed in the square  $[-1, 1]^2$  and touches its sides at the points  $(1, 0)$  and  $(0, 1)$ , the assumption  $\xi_0 \geq \eta_0 \geq 0$  implies that  $\xi_0 \geq \frac{1}{2}$  and the function  $L_0$  has the form  $L_0(x, y) = ax + by$  where

$$a = L_0(1, 0) \in (0, 1]$$

and

$$b = L_0(0, 1) \in [0, 1).$$

Moreover the coefficient  $a$  is bounded from below by a constant  $a_0 > 0$  determined by the shape of  $B$ , since the only supporting functions vanishing at  $(1, 0)$  are those at the points  $(0, \pm 1) \in S$ , and these points are separated away from the range  $\{\xi_0 \geq \eta_0 \geq 0\}$  that we restrict ourselves to.

Differentiating the definition of  $f_\sigma$ :

$$f_\sigma(x, y) = (1 - \sigma^2 x^2 - \sigma^2 y^2) \cdot (x - \sigma x^3, y - \sigma y^3)$$

yields

$$\begin{aligned} \frac{\partial f_\sigma}{\partial x}(x, y) &= (1 - \sigma^2 x^2 - \sigma^2 y^2) \cdot (1 - 3\sigma x^2, 0) - 2\sigma^2 x(x - \sigma x^3, y - \sigma y^3) \\ &= (1 - A_{11}, -A_{21}) \end{aligned}$$

where

$$A_{11} = 3\sigma x^2 + \sigma^2 x^2(3 - 5\sigma x^2) + \sigma^2 y^2(1 - 3\sigma x^2),$$

$$A_{21} = 2\sigma^2 xy(1 - \sigma y^2)$$

and, similarly,

$$\frac{\partial f_\sigma}{\partial y}(x, y) = (-A_{12}, 1 - A_{22})$$

where

$$A_{12} = 2\sigma^2 xy(1 - \sigma x^2),$$

$$A_{22} = 3\sigma y^2 + \sigma^2 y^2(3 - 5\sigma y^2) + \sigma^2 x^2(1 - 3\sigma y^2).$$

Now for every  $(\xi, \eta) \in S$  we have

$$df_\sigma(x, y, \xi, \eta) = (\xi, \eta) - (A_1, A_2)$$

where

$$A_1 = \xi A_{11} + \eta A_{12}, \quad A_2 = \xi A_{21} + \eta A_{22}$$

and hence

$$(3.4) \quad \begin{aligned} L_0(df_\sigma(x, y, \xi, \eta)) &= L_0(\xi, \eta) - L_0(A_1, A_2) \\ &\leq 1 - c_0(\xi - \xi_0)^2 - c_0(\eta - \eta_0)^2 - L_0(A_1, A_2) \end{aligned}$$

by (3.2). If  $(\xi, \eta)$  is separated away from  $(\xi_0, \eta_0)$  by a constant (e.g. by  $\frac{1}{10}$ ), this inequality implies (3.3), since  $A_{ij}$  are small when  $\sigma$ ,  $|x|$  and  $|y|$  are small. Thus we may assume that  $(\xi, \eta)$  is  $\frac{1}{10}$ -close to  $(\xi_0, \eta_0)$  and therefore  $\xi \geq \frac{1}{3}$ . We need to estimate from below the term  $L_0(A_1, A_2)$  in (3.4). Recall that

$$(3.5) \quad L_0(A_1, A_2) = aA_1 + bA_2 = a\xi A_{11} + a\eta A_{12} + b\xi A_{21} + b\eta A_{22}.$$

Assuming  $\sigma, |x|, |y| < \frac{1}{10}$ , we estimate

$$(3.6) \quad \begin{aligned} |a\eta A_{12}| &\leq |A_{12}| \leq \sigma^2 xy, \\ |b\xi A_{21}| &\leq |A_{21}| \leq \sigma^2 xy \end{aligned}$$

(since  $|a|, |b|, |\xi|, |\eta| \leq 1$ ), and

$$A_{11} \geq 3\sigma x^2 + \frac{2}{3}\sigma^2 y^2 \geq \sigma x^2 + \frac{1}{6}\sigma^2 y^2 + 2\sigma^{3/2}xy$$

where the last inequality follows from the Cauchy inequality applied to  $2\sigma x^2$  and  $\frac{1}{2}\sigma^2 y^2$ , namely  $2\sigma x^2 + \frac{1}{2}\sigma^2 y^2 \geq 2\sigma^{3/2}xy$ . Since  $a \geq a_0$ ,  $\xi \geq \frac{1}{3}$ , and

$$2\sigma^{3/2}xy = \sigma^{-1/2}(2\sigma^2 xy) \geq \sigma^{-1/2}|a\eta A_{12} + b\xi A_{21}|$$

by (3.6), it follows that

$$a\xi A_{11} \geq c_1\sigma^2(x^2 + y^2) + c_2\sigma^{-1/2}|a\eta A_{12} + b\xi A_{21}|$$

where  $c_1 = a_0/18$  and  $c_2 = a_0/3$ . Assuming  $\sigma < c_2^{-2}$ , it follows that

$$a\xi A_{11} + a\eta A_{12} + b\xi A_{21} \geq c_1\sigma^2(x^2 + y^2).$$

This and (3.5) imply that

$$L_0(A_1, A_2) \geq c_1\sigma^2(x^2 + y^2) + b\eta A_{22}$$

and then from (3.4) we have

$$(3.7) \quad L_0(df_\sigma(x, y, \xi, \eta)) \leq 1 - c_0(\xi - \xi_0)^2 - c_0(\eta - \eta_0)^2 - c_1\sigma^2(x^2 + y^2) - b\eta A_{22}.$$

To achieve our goal (3.3), it suffices to get rid of the last term  $b\eta A_{22}$ . Observe that  $A_{22} \geq 0$ . Therefore in the case  $\eta \geq 0$  we have  $b\eta A_{22} \geq 0$  and the result follows. It remains to consider the case  $\eta \leq 0$ . Observe that  $A_{22} \leq 4\sigma(x^2 + y^2)$ , therefore

$$(3.8) \quad |b\eta A_{22}| \leq 4|\eta|\sigma(x^2 + y^2).$$

In the case  $|\eta| < c_1\sigma/10$ , this implies that

$$|b\eta A_{22}| \leq \frac{1}{2}c_1\sigma^2(x^2 + y^2),$$

so the term  $b\eta A_{22}$  in (3.7) is majorized by the term  $c_1\sigma^2(x^2 + y^2)$ . And in the case  $|\eta| \geq c_1\sigma/10$ , the fact that  $\eta \leq 0 \leq \eta_0$  implies

$$c_0(\eta - \eta_0)^2 \geq c_0\eta^2 \geq c_3\sigma^2$$

where  $c_3 = c_0 c_1^2 / 100$ . Recall that  $|x|, |y| \ll \sigma$  (we are choosing  $U$  after  $\sigma$ ), so we may assume that  $x^2 + y^2 < c_3 \sigma / 10$ . Then (3.8) implies that

$$|b\eta A_{22}| \leq \frac{1}{2} c_3 \sigma^2 \leq \frac{1}{2} c_0 (\eta - \eta_0)^2,$$

so the term  $b\eta A_{22}$  in (3.7) is majorized by the term  $c_0(\eta - \eta_0)^2$ .

Thus we have proved (3.3). This finishes the proof of Lemma 3.5.  $\square$

**3.3. General case.** Since every metric is close to a constant one in a neighborhood of the origin, Lemma 3.5 easily generalizes to arbitrary Finsler metrics on the plane. Namely the following holds.

**Lemma 3.6.** *Let  $\varphi$  be a Finsler metric on  $\mathbb{R}^2$ . Then there exist a neighborhood  $U \subset \mathbb{R}^2$  of the origin and a smooth saddle embedding  $F: U \rightarrow \mathbb{R}^4$  such that the map  $dF|_{S_\varphi U}$  is an embedding and its image  $dF(S_\varphi U)$  is a pre-convex surface in  $\mathbb{R}^4$ .*

*Proof.* Let  $\varphi_0 = \varphi|_{T_0 \mathbb{R}^2}$ . We also consider  $\varphi_0$  as a constant Finsler metric on  $\mathbb{R}^2$ . For every  $\varepsilon > 0$ , define a “blow-up” metric  $\varphi_\varepsilon$  on  $\mathbb{R}^2$  defined by

$$\varphi_\varepsilon(x, v) = \varphi(\varepsilon^{-1}x, v), \quad x \in \mathbb{R}^2, \quad v \in T_x \mathbb{R}^2.$$

Note that  $\varphi_\varepsilon$  converge to  $\varphi_0$  with all derivatives on compact sets as  $\varepsilon \rightarrow 0$ .

By Lemma 3.5, there is a neighborhood  $U \subset \mathbb{R}^2$  of the origin and a strictly saddle embedding  $F: U \rightarrow \mathbb{R}^4$  such that  $\Sigma_0 := dF(S_{\varphi_0} U)$  is a pre-convex surface in  $\mathbb{R}^4$ . Fix a neighborhood  $U' \Subset U$  of the origin. Since the surfaces  $\Sigma_\varepsilon := dF(S_{\varphi_\varepsilon} U)$  converge to  $\Sigma_0$  with all derivatives on compact sets as  $\varepsilon \rightarrow 0$ , the smaller surfaces  $\Sigma'_\varepsilon := dF(S_{\varphi_\varepsilon} U')$  are pre-convex for all sufficiently small  $\varepsilon > 0$ . Fix such an  $\varepsilon$  and observe that the map

$$F_\varepsilon : x \mapsto \varepsilon^{-1} F(\varepsilon^{-1} x),$$

from the neighborhood  $\varepsilon U'$  of the origin to  $\mathbb{R}^4$ , parameterizes a surface homothetic to  $F$  in  $\mathbb{R}^4$  (and hence is strictly saddle) and  $dF_\varepsilon(S_{\varphi_\varepsilon}(\varepsilon^{-1} U')) = \Sigma'_\varepsilon$ . Thus  $F_\varepsilon$  and  $\varepsilon U'$  suit for  $F$  and  $U$  from the statement of the lemma.  $\square$

Now we are in position to prove Theorem 1.4. Since the statement of the theorem is local, it suffices to prove it for  $M = (\mathbb{R}^2, \varphi)$  and  $x = 0$  where  $\varphi$  is a Finsler metric on  $\mathbb{R}^2$ . By Lemma 3.6, there is a neighborhood  $U \subset \mathbb{R}^2$  of the origin and a smooth saddle embedding  $F: U \rightarrow \mathbb{R}^4$  such that the map  $dF|_{S_\varphi U}$  parameterizes a pre-convex hypersurface  $\Sigma \subset \mathbb{R}^4$ . Note that  $\Sigma$  is symmetric with respect to the origin.

By Lemma 3.4, there exists a symmetric, compact, smooth, quadratically convex surface  $\Sigma' \subset \mathbb{R}^4$  which contains a neighborhood  $U_0$  of the set  $K = dF(S_0) \subset \Sigma$  where  $S_0$  is the unit sphere of  $\varphi$  in  $T_0 \mathbb{R}^2$ . This surface is the unit sphere of some smooth and quadratically convex norm  $\|\cdot\|$  on  $\mathbb{R}^4$ .

For a sufficiently small neighbourhood  $U' \subset U$  of 0, we have  $dF(S_\varphi U') \subset U_0 \subset \Sigma'$ . Therefore  $\|dF(x, v)\| = 1$  for every  $x \in U'$  and every  $\varphi$ -unit vector  $v \in T_x \mathbb{R}^2$ . This means that  $F$  is an isometric embedding of  $(U', \varphi)$  to  $(\mathbb{R}^4, \|\cdot\|)$ . This finishes the proof of Theorem 1.4.

#### 4. COMPLETE CONVEX SURFACES

The goal of this section is to prove Theorem 1.7. Our plan is the following. Assuming that there is a geodesic line on a surface of a convex set  $B$  in a 3-dimensional normed space  $V$ , we rescale  $B$  with coefficients going to zero and pass to the limit. This yields a geodesic line on the surface of the asymptotic cone of  $B$ ,

and this geodesic line contains the cone's apex. However on a surface of a sharp convex cone no shortest path can pass through the apex, as shown in Lemma 4.2.

A straightforward realization of this plan would require us to prove that intrinsic metrics of converging convex surfaces converge to the intrinsic metric of their limit (which is not necessarily smooth). While this fact is standard in Euclidean spaces and certainly true in general normed spaces, we do not know an elegant proof and do not want to mess with a cumbersome one here. We work around this issue by constructing shortcut paths lying in planar sections of our surfaces (and for planar convex curves the convergence of lengths is easy, see Lemma 4.1).

**Notation.** For a vector space  $V$  and points  $p_1, p_2, \dots, p_n \in V$ , we denote by  $[p_1, p_2, \dots, p_n]$  the broken line composed of segments  $[p_i p_{i+1}]$ ,  $i = 1, \dots, n-1$ . If  $V$  is equipped with a norm  $\|\cdot\|$ , the length of this broken line is given by

$$\text{length}[p_1, p_2, \dots, p_n] = \sum_{i=1}^{n-1} \|p_i - p_{i+1}\|.$$

We need the following standard fact about perimeters of two-dimensional convex sets (supplied with a proof for the sake of completeness).

**Lemma 4.1.** *Let  $V$  be a two-dimensional normed space and  $B \subset V$  a compact convex set with nonempty interior. Then*

1. *For every compact convex set  $B' \supset B$  one has  $\text{length}(\partial B) \leq \text{length}(\partial B')$ .*
2. *If  $\{B_i\}$  is a sequence of convex sets in  $V$  converging to  $B$  (in the Hausdorff metric), then  $\text{length}(\partial B_i) \rightarrow \text{length}(\partial B)$ .*

*Proof.* 1. Since the length of  $\partial B$  is a limit of lengths of inscribed polygons, it suffices to prove the lemma in the case when  $B$  is a polygon. Let  $\partial B = [p_1, p_2, \dots, p_n, p_1]$ . If we cut  $B'$  along a line containing a segment  $[p_i p_{i+1}]$  and remove the piece that does not contain  $B$ , the perimeter of  $B'$  can only get smaller, by the triangle inequality. Thus we can make  $B$  from  $B'$  by finitely many operations each of which does not increase the perimeter. Hence  $\text{length}(\partial B) \leq \text{length}(\partial B')$ .

2. Choose the origin in the interior of  $B$ . Then the assumption that  $B_i \rightarrow B$  is equivalent to the following:

$$(1 - \varepsilon_i)B \subset B_i \subset (1 + \varepsilon_i)B$$

for some sequence  $\varepsilon_i \rightarrow 0$ . By the first part of the lemma, this implies that

$$(1 - \varepsilon_i) \text{length}(\partial B) \subset \text{length}(\partial B_i) \subset (1 + \varepsilon_i) \text{length}(\partial B),$$

hence the result.  $\square$

**Lemma 4.2.** *Let  $V$  be a 3-dimensional normed space whose norm is  $C^1$ -smooth and strictly convex. Let  $K \subset V$  be a sharp cone. Then for every two points  $p, q \in \partial K \setminus \{0\}$  there exists a path that connects  $p$  and  $q$  in  $\partial K$ , is strictly shorter than the broken line  $[p, 0, q]$ , and is contained in some plane  $\alpha \subset V$ .*

*Proof.* Let  $H_1$  and  $H_2$  be supporting planes to  $K$  at  $p$  and  $q$  respectively. Since the cone is sharp, there is a third supporting plane  $H_3$  that does not contain the intersection line  $H_1 \cap H_2$ . Consider the trihedral cone  $K' = H_1^+ \cap H_2^+ \cap H_3^+$  where  $H_i^+$  denotes the half-space bounded by  $H_i$  and containing  $K$ .

It suffices to prove the lemma for  $K'$  in place of  $K$ . Indeed, suppose that for some plane  $\alpha \subset V$  a boundary arc  $\sigma'$  of  $F' := \alpha \cap K'$  between  $p$  and  $q$  is shorter



than  $[p, 0, q]$ . Consider the corresponding (that is, lying in the same half-plane with respect to the line  $\langle pq \rangle \subset \alpha$ ) boundary arc  $\sigma$  of  $F := \alpha \cap K$ . Since  $F \subset F'$ , Lemma 4.1 implies that

$$\text{length}(\sigma) \leq \text{length}(\sigma') < \text{length}[p, 0, q]$$

and the lemma follows from its restatement for  $K'$ .

Thus now we restrict ourselves to proving the assertion for  $K'$ .

Let  $v$  be a nonzero vector in the line  $H_1 \cap H_2$  pointing outwards  $K'$  (that is,  $-v$  points in the direction of an edge of  $K'$ ). Define

$$f(t) = \text{length}[p, vt, q] = \|p - vt\| + \|q - vt\|.$$

Note that  $f$  is a strictly convex function differentiable at 0.

If  $f'(0) > 0$ , then  $f(-t) < f(0)$  for a small  $t > 0$ . Observe that  $f(-t)$  is the length of the broken line  $[p, -vt, q]$  which lies on  $\partial K'$  and is contained in a plane (since it has only two edges). Thus we have found a desired broken line in the case when  $f'(0) > 0$ .

It remains to consider the case when  $f'(0) \leq 0$ . For every  $t \geq 0$ , let  $a(t)$  and  $b(t)$  denote the intersection points of segments  $[p, vt]$  and  $[q, vt]$  with the plane  $H_3$ . Note that  $a(t)$  and  $b(t)$  lie on edges of  $K'$  and the broken line  $[p, a(t), b(t), q]$  is contained in  $\partial K'$ . For  $t = 0$ , we have  $a(0) = b(0) = 0$ .

One easily sees that  $a(t)$  and  $b(t)$  are differentiable in  $t$  and their derivatives at 0 are nonzero vectors (pointing in the directions of the respective edges). Denote these vectors by  $v_1$  and  $v_2$  and define

$$g(t) = \text{length}[p, a(t), b(t), q].$$

Then

$$f(t) - g(t) = \|vt - a(t)\| + \|vt - b(t)\| - \|a(t) - b(t)\|.$$

Therefore

$$\lim_{t \rightarrow +0} \frac{f(t) - g(t)}{t} = \|v - v_1\| + \|v - v_2\| - \|v_1 - v_2\| > 0$$

by the strict triangle inequality for the norm  $\|\cdot\|$ . Hence

$$\lim_{t \rightarrow +0} \frac{g(t) - f(0)}{t} = f'(0) - \lim_{t \rightarrow +0} \frac{f(t) - g(t)}{t} < 0$$

since  $f'(0) \leq 0$ . Therefore  $g(t) < f(0)$  for all sufficiently small  $t > 0$ . Thus, for a small  $t > 0$ , the broken line  $[p, a(t), b(t), q]$  is shorter than  $[p, 0, q]$ . By construction, this broken line lies in the plane through the points  $p$ ,  $q$  and  $vt$ .  $\square$

*Proof of Theorem 1.7.* We may assume that the origin is contained in the interior of  $B$ . Suppose that there is a geodesic line  $\gamma: (-\infty, \infty) \rightarrow \partial B$ . For every  $\lambda > 1$ , let  $H^\lambda: V \rightarrow V$  denote the homothety with coefficient  $\lambda^{-1}$ , that is,  $H^\lambda(x) = \lambda^{-1}x$  for all  $x \in V$ . Let  $B^\lambda = H^\lambda(B)$  and  $\gamma^\lambda: [-1, 1] \rightarrow \partial B^\lambda$  is a path defined by  $\gamma^\lambda(t) = H^\lambda(\lambda t)$ . Note that  $\gamma^\lambda$  is a homothetic image of  $\gamma|_{[-\lambda, \lambda]}$  reparameterized by arc length. Since  $\gamma$  is a geodesic line on  $\partial B$ ,  $\gamma^\lambda$  is a shortest path on  $\partial B^\lambda$ .

Now let  $\lambda \rightarrow \infty$ . The sets  $B^\lambda$  converge to the asymptotic cone  $K := \bigcap_{\lambda > 1} B^\lambda$ . Since  $B$  does not contain straight lines,  $K$  is a sharp cone. We assume that  $K$  has nonempty interior (the case when  $K$  is contained in a plane is similar and left to the reader). Therefore the endpoints of the curves  $\gamma^\lambda$  lie within a compact region in  $V$ . Choose a subsequence  $\{\lambda_i\}$ ,  $\lambda_i \rightarrow \infty$ , such that  $p_i := \gamma^{\lambda_i}(-1)$  and  $q_i := \gamma^{\lambda_i}(1)$  converge to some points  $p, q \in \partial K$ . Since the curves  $\gamma^{\lambda_i}$  are 1-Lipschitz

and  $\gamma^{\lambda_i}(0) = \lambda_i^{-1}\gamma(0) \rightarrow 0$ , the distances from  $p$  and  $q$  to the origin are not greater than 1. Therefore by Lemma 4.2 there is a plane  $\alpha \subset V$  such that a boundary arc  $\sigma$  of  $\alpha \cap K$  between  $p$  and  $q$  has  $\text{length}(\sigma) < 2$ .

We assume that  $p \neq q$  (the case  $p = q$  is trivial). Fix a point  $o \in \alpha \cap \text{int}(K)$ . For each  $i \gg 1$ , let  $\alpha_i \subset V$  be the plane through  $o$ ,  $p_i$  and  $q_i$ . Note that these planes converge to  $\alpha$ , hence there are boundary arcs  $\sigma_i$  of  $\alpha_i \cap B^{\lambda_i}$  that converge to  $\sigma$ . Consider a “triangle”  $T \subset \alpha$  bounded by  $\sigma$  and the segments  $[op]$ ,  $[oq]$ . Applying Lemma 4.1 to  $T$  and suitable projections of corresponding “triangles” in the planes  $\alpha_i$  (and taking into account that the norms on  $\alpha_i$  Lipschitz converge to the norm on  $\alpha$ ) yields that  $\text{length}(\sigma_i) \rightarrow \text{length}(\sigma) < 2$ . Hence  $\text{length}(\sigma_i) < 2 = \text{length}(\gamma^{\lambda_i})$  for a sufficiently large  $i$ . Therefore  $\gamma^{\lambda_i}$  is not a shortest path on  $\partial B^{\lambda_i}$ , a contradiction.  $\square$

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